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A REDUCTION METHOD FOR LARGE SCALE UNSYMMETRIC EIGENVALUE PROBLEMS IN STRUCTURAL DYNAMICS

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The discussion begins with the classification of eigenvalue problems arising from conservative and non-conservative structural systems. The conservative type includes undamped structural eigenvalue problems and undamped gyroscopic eigenvalue problems. The non-conservative type includes damped structural eigenvalue problems, damped gyroscopic eigenvalue problems and constrainedly damped eigenvalue problems. The methods for solving large scale unsymmetric eigenvalue problems are briefly reviewed. The advantages and properties of Arnoldi's method have also been discussed. Arnoldi's reduction method has been generalized and the partial solution of large scale unsymmetric-definite eigenvalue problems in structural dynamics is presented in detail. A very simple reduction algorithm is obtained by simplifying the proposed method for undamped gyroscopic eigenvalue problems. To make the proposed reduction method feasible for engineering problems, a restart technique is introduced to work with Arnoldi's reduction method for checking and computing missing eigenvalues. Numerical examples are also presented to demonstrate the effectiveness of the proposed reduction method.

1. INTRODUCTION

An eigenvalue problem for modern structural dynamic analysis can be generally expressed as follows [1] (a list of nomenclature is given in Appendix B):

$$\lambda^2 Mx + \lambda(G + C)x + (K + D)x = 0. \tag{1}$$

If the gyroscopic, damping and constraint damping matrices G, C and D are not present, this eigenvalue problem simplifies to an undamped structural eigenvalue problem:

$$\lambda^2 Mx + Kx = 0. \tag{2}$$

Equation (2) is frequently encountered in engineering analysis. However, a practical structure is always damped. If the damping is light, the system is usually assumed to have proportional damping, so that the problem still can be solved as easily as eigenvalue problem (2). In modern structures, because of the application of new energy dissipative

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materials and concentrated damping devices, engineers will face the following quadratic eigenvalue problem:

$$\lambda^2 Mx + \lambda Cx + Kx = 0. \tag{3}$$

Furthermore, in the dynamic analysis of rotating structures, such as helicopter blades and rotating stabilized satellites, where Coriolis forces may have important effects on dynamic characteristics of the whole system, the following undamped and damped gyroscopic eigenvalue problems, respectively, need to be solved:

$$\lambda^2 Mx + \lambda Gx + Kx = 0, \qquad \lambda^2 Mx + \lambda(G + C)x + Kx = 0. \tag{4, 5}$$

If the system is forced by the follower forces, the linearized equation about its equilibrium will generally result in a symmetric stiffness matrix K and a skew-symmetric constraint damping matrix D. In such cases, the eigenvalue problem will take the form of equation (1). The constraint damping may also appear in the governing equation in other cases such as the stability of plasma [2–6].

With finite element modelling (FEM), the size of the eigenvalue problems (1)–(5) is usually very large, but the system matrices M, G, C, K and D are usually sparsely populated. For most engineering eigenvalue problems, one is interested only in the lower eigenvalues and corresponding modes which need to be investigated. Hence a method for large scale eigenvalue problems should have the capability reliably to find those lower eigenvalues while fully utilizing the sparsity of the system matrices. Reduction methods, such as the Lanczos method and the subspace iteration method, are appropriate for such uses.

The generalized symmetric eigenvalue problem (2) can be efficiently solved by the subspace iteration method [7] and the symmetric Lanczos method [8–10]. The undamped gyroscopic eigenvalue problem has been under study for many years. Currently, Meirovitch's method [11], Bauchau's Lanczos method [12] and Gupta's block Lanczos method [13] are available. Since Coriolis forces do no work in any virtual displacement, the gyroscopic eigenvalue problem can be classified as of the same type as the undamped structural eigenvalue problem described in equation (2), belonging to the conservative type. This classification is assured by the fact that both two types of eigenvalue problems take on a skew-symmetric form in the state space. The symmetry of the gyroscopic eigenvalue problem (4) and the undamped structural eigenvalue problem (2) are derived from the conservative nature of structural systems. Rewriting equation (1) in the state space, one has

$$\lambda \begin{bmatrix} -(G + C) & -M \\ M & 0 \end{bmatrix} \begin{Bmatrix} x \\ \lambda x \end{Bmatrix} = \begin{bmatrix} K + D & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} x \\ \lambda x \end{Bmatrix}. \tag{6}$$

In terms of the quantities

$$A = \begin{bmatrix} -(G + C) & -M \\ M & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} K + D & 0 \\ 0 & M \end{bmatrix}, \qquad y = \begin{bmatrix} x \\ \lambda x \end{bmatrix},$$

equation (6) becomes

$$\lambda Ay = By. \tag{7}$$

The presence of the damping matrix C and the constraint damping matrix D in equation (6) has destroyed the symmetry of the system matrices. From the physical point of view, the damping matrix C and the circulatory matrix D are due to the non-conservation of

energy in the system. In such cases, the eigenvalue problem (1) becomes materially unsymmetric.

The difference between conservative and non-conservative eigenvalue problems may result in very different phenomena in the procedure of finding solutions. For example, if the two-sided Lanczos method [14] is applied to the unsymmetric eigenvalue problems (3) and (5), there will be a possibility of serious breakdown [14], as in the methods proposed in references [15–17]. However, this phenomenon vanishes when the two-sided Lanczos method is applied to the conservative type of eigenvalue problems (2) and (4), as in Bauchau's gyroscopic Lanczos method [12] and Gupta's block gyroscopic Lanczos method [13]. If the two-sided Lanczos method is applied to a non-conservative or unsymmetric eigenvalue problem, complex computation cannot be avoided; while for the conservative eigenvalue problem, complex computation is unnecessary.

Arnoldi's method [18] was developed shortly after Lanczos' method [8]. With Arnoldi's method, there is no possibility of serious breakdown for both the symmetric and unsymmetric matrices, and the algorithm is conducted entirely in the real domain for any real matrices. However, Arnoldi's method is less well known than Lanczos' method because, for a complete reduction of a matrix to its upper Hessenberg form by Arnoldi's method, the computational cost is higher than by Housholder's method, due to additional re-orthogonalization requirements [14]. For modern large scale sparse eigenvalue problems, if Arnoldi's method is used to determine a few end eigenvalues, it may be very efficient. In recent years, interest has been shown by mathematicians [19–21].

The main purpose of this paper is to develop a reduction technique based on Arnoldi's method for the eigensolution of large scale unsymmetric eigenvalue problems in structural dynamics. The method should be capable of finding all the lower eigenvalues reliably without missing any. Meanwhile, it should be robust, and it also should avoid complex computation and take advantage of the sparsity of system matrices.

2. AN ARNOLDI ALGORITHM FOR UNSYMMETRIC EIGENVALUE PROBLEMS IN STRUCTURAL DYNAMICS

The basic Arnoldi process for a standard matrix can be found in reference [18]. An Arnoldi reduction for the generalized eigenvalue problem (7) is defined in Algorithm 1 below. There is no loss of generality in assuming that matrix B is non-singular. A very general shift technique can be found from reference [22] for transforming the quadratic eigenvalue problem to that of a single matrix in case matrix B is singular. The following transformation is adopted from reference [22]: $\mu = (\theta + \lambda)/(\theta - \lambda)$. By choosing a positive number for the parameter μ, the transformation has very favorable properties for studying the stability of the system according to reference [22]. As an alternative, the following simple shift technique can also be accepted. If B is singular, a shift λ_0 can be made in equation (1) by letting $\lambda = \mu + \lambda_0$; then equation (1) is transformed to

$$\mu^2 M x + \mu(G + \tilde{C})x + (\tilde{K} + \tilde{D})x = 0, \tag{1a}$$

where $\tilde{C} = C + 2\lambda_0 M = \tilde{C}^{\mathrm{T}}$, $\tilde{K} = K + \lambda_0 C + \lambda_0^2 M = \tilde{K}^{\mathrm{T}}$ and $\tilde{D} = D + \lambda_0 G = -\tilde{D}^{\mathrm{T}}$. Using the finite element method to establish the system governing equation, all system matrices generally have the same sparsity structure. Carrying out the shift with equation (1) instead of equation (7) not only retains the sparsity of the system matrices but also the structure of matrices A and B.

Algorithm 1: generalized Arnoldi reduction process for unsymmetric eigenvalue problem (7)

Step 1: specify u_1 as an arbitrary vector satisfying $\|u_1\|_2 = 1$;

Step 2: generate new Arnoldi vectors from the following recurrence equations:

$$\beta_{k+1}u_{k+1} = r_k = B^{-1}Au_k - \sum_{i=1}^{k} h_{ik}u_i, \qquad k = 1, \ldots, \tag{8}$$

$$h_{ik} = u_i^{\mathrm{T}}B^{-1}Au_k, \quad i = 1, \ldots, k, \qquad \beta_{k+1} = \|r_k\|_2, \tag{9a, b}$$

where h_{ik} makes u_{k+1} orthogonalized to u_i $(i = 1, \ldots, k)$.

Algorithm 1 is in a form suitable for obtaining the reciprocals of lower eigenvalues of equation (7). If there is no breakdown in the reduction process, $u_1, \ldots, u_{k+1}$ are a set of orthogonalized bases in the Krylov subspace $\{u_1, B^{-1}Au_1, \ldots, (B^{-1}A)^k u_1\}$. Theoretically, this algorithm will halt. In this respect, the following theorems are relevant.

Theorem 1. Assume that the eigenvectors of problem (7) are complete. If u_1 can be expressed as a linear combination of the eigenvectors y_i $(i = 1, \ldots, m, m < n)$ with respect to different eigenvalues λ_i $(i = 1, \ldots, m)$, i.e.,

$$u_1 = \sum_{i=1}^{m} s_i y_i, \qquad s_i \neq 0, \quad i = 1, \ldots, m,$$

$$\lambda_i \neq \lambda_j, \qquad i \neq j,$$

the Arnoldi Algorithm 1 can be executed for only m steps, at which time r_m must be zero.

Theorem 2. Assume that eigenvalue problem (7) is complete in eigenvectors, and that m is the number of its non-equal eigenvalue (equal eigenvalues are counted as one). Then, with any start vectors u_1, the Arnoldi Algorithm 1 will break down before step $(m + 1)$.

The proofs of Theorems 1 and 2 are given in Appendix A.

Fortunately, Wilkinson [14] has pointed out that this type of breakdown can be overcome by arbitrarily taking a unit vector $u_{k+1} \in R^{2n \times 1}$ which is orthogonalized to the previously generated Arnoldi vectors, and letting $\beta_{k+1} = 0$: i.e., the Arnoldi process can be continued after the breakdown, and the breakdown of the Arnoldi process means that the Krylov subspace generated is an invariant subspace. In finite precision computation, the breakdown rarely happens. Instead, the orthogonality is lost among the Arnoldi vectors because of "cancellation" other than the rounding errors, as emphasized by Wilkinson [14]. The orthogonality among the Arnoldi vectors can be restored by a complete re-orthogonalization [14, 9]. Based on the floating point expression, an explanation has been given on the mechanism of loss of the orthogonality and on why one re-orthogonalization can restore the orthogonality [23]. In the time of Wilkinson, Arnoldi's method was used to reduce a matrix to its upper Hessenberg form, and the complete re-orthogonalization made it less attractive in finite precision computation as compared with Householder's method. However, Householder's method destroys the sparsity of the original matrices in the reduction process. This destruction of sparsity prevents it from being a practical method for large scale eigenvalue problems in engineering, whereas the sparsity of system matrices can be fully exploited in Arnoldi's process. In addition, the property of the Krylov subspace makes it unnecessary to carry out Arnoldi's process in full scale in order that the required end eigenvalues converge. In view of the great success of the Lanczos method in large scale symmetric eigenvalue problems, it is reasonable to believe that Arnoldi's method will become a successful method for large scale unsymmetric eigenvalue problems. Usually, the number of Arnoldi vectors which need to be generated to enable a few of the end eigenvalues converge is much less than the size of the original matrix. Even if complete re-orthogonalization is enforced in the finite precision

implementation of Arnoldi's method, it remains an efficient method compared with any methods that destroy the sparsity for a large scale problem. A selective re-orthogonalization technique can also be developed as a parallel with the selective re-orthogonalization technique for symmetric eigenvalue problems [10]. With selective re-orthogonalization, the newly generated Lanczos vector is re-orthogonalized with respect to the good Ritz vectors instead of the generated Lanczos vectors in order to keep the orthogonality among the Lanczos vectors in finite precision computation. The Daniel–Paige theory is the basis of the selective re-orthogonalization, which tells one that the newly generated Lanczos vectors tend to have a non-trivial component in the direction of any converged Ritz vectors in finite precision. The Ritz vectors are defined as approximations of the eigenvectors of the original matrix based on the eigensolution of the projected system on the generated Krylov subspace. Refer to reference [10] for the definition of good Ritz vectors. There are other variations in the re-orthogonalizations [19]. In the authors' experience, selective re-orthogonalization may not be necessarily less expensive than complete re-orthogonalization due to the computation of the Ritz vectors, but it gives information about the convergence of the Ritz vectors.

Theoretically, after $2n$ steps of the Arnoldi process, u_{2n+1} must be zero. In finite precision computation, u_{2n+1} is zero within the round-off error of the finite precision if the re-orthogonalization is enforced, giving

$$B^{-1}AU = UH, \tag{10}$$

where

$$U = [u_1, \ldots, u_{2n}], \qquad H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1,2n} \\ \beta_2 & h_{22} & \cdots & h_{2,2n} \\ & \cdots & \cdots & \cdots \\ & & \beta_{2n} & h_{2n,2n} \end{bmatrix}. \tag{11, 12}$$

U is an orthogonal matrix. Therefore, solving problem (1) can be performed by determining the upper Hessenberg matrix H and its eigensolution. If the system is constraint damped, the whole eigensolution of H is needed to determine the stability of the system. However, for a system without constraint damping and in which B is definite, there is no problem of stability. The size of the eigenvalue problem is usually very large and it is unnecessarily expensive to find all the eigenvalues.

3. A REDUCTION METHOD FOR UNSYMMETRIC-DEFINITE PROBLEMS

In this section an Arnoldi algorithm suitable for damped modal analysis in structural dynamics is discussed. Consider eigenvalue problem (1) without matrix D, which includes problems (2)–(5). In the following, matrix B is assumed to be positive definite; the following generalized Arnoldi process for the unsymmetric-definite problem may be used.

Algorithm 2: a generalized Arnoldi process for the unsymmetric-definite problem

Step 1: let u_1 be an arbitrary vector satisfying $\|u_1\| = (u_1^{\mathrm{T}} B u_1)^{1/2} = 1$;

Step 2: new Arnoldi vectors are generated by

$$\beta_{k+1} u_{k+1} = r_k = B^{-1} A u_k - \sum_{i=1}^{k} h_{ik} u_i, \qquad k = 1, \ldots, \tag{13}$$

$$h_{ik} = u_i^{\mathrm{T}} A u_k, \quad i = 1, \ldots, k, \qquad \beta_{k+1} = (r_k^{\mathrm{T}} B r_k)^{1/2}, \tag{14, 15}$$

where h_{ik} makes u_{k+1} B-orthogonal to u_i: i.e., $u_{k+1}^{\mathrm{T}} B u_i = 0$, $i = 1, \ldots, k$.

By keeping B-orthogonalities among Arnoldi vectors, the symmetric property of matrix A can be retained in the projected upper Hessenberg matrix H_k: i.e., if matrix A is symmetric, H_k is symmetric; if matrix A is skew-symmetric, H_k is also skew-symmetric; and if matrix A is unsymmetric, H_k is also unsymmetric. Through the symmetry of matrix H_k, the reduction process can be simplified and the Arnoldi process can directly degenerate to the Lanczos process for the symmetric problem. Furthermore, the quadratic form about matrix B is the expression of system energy; hence normalizing the Arnoldi vectors with the quadratic form about matrix B has also a physical background.

For any $k \geqslant 2$, one has

$$B^{-1}AU_k = U_kH_k + \beta_{k+1}u_{k+1}e^{\mathrm{T}}, \tag{16}$$

$$U_k^{\mathrm{T}}AU_k = H_k, \qquad H_k = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ \beta_2 & h_{22} & \cdots & h_{2k} \\ & \cdots & \cdots & \cdots \\ & & \beta_k & h_{kk} \end{bmatrix}. \tag{17, 18}$$

Obtain the complete eigensolution of the upper Hessenberg matrix H_k, i.e.,

$$H_kS_k = S_k\Lambda_k, \tag{19}$$

where

$$\Lambda_k = \mathrm{diag}\,[\lambda_1, \ldots, \lambda_k], \qquad S_k = [s_1, \ldots, s_k] \tag{20a, b}$$

are, respectively, the eigenvalue matrix and its corresponding right eigenvector matrix. By writing down equation (19), it is tacitly assumed that the eigenvectors of matrix H_k are complete. From equations (16) and (19),

$$B^{-1}A\{U_ks_i\} - \lambda_i\{U_ks_i\} = \beta_{k+1}s_{ki}u_{k+1}, \tag{21a}$$

$$\|B^{-1}A\{U_ks_i\} - \lambda_i\{U_ks_i\}\| = \beta_{k+1}|s_{ki}|\,\|u_{k+1}\|. \tag{21b}$$

This indicates that if $\beta_{k+1}|s_{ki}|\,\|u_{k+1}\|$ is small enough, denoted as

$$\beta_{k+1}|s_{ki}|\,\|u_{k+1}\| < \varepsilon, \tag{22}$$

$\{\lambda_i, U_ks_i\}$ will be an approximate eigenpair of $B^{-1}A$. Therefore, an Arnoldi reduction method for unsymmetric-definite eigenvalue problems can be summarized as follows.

Algorithm 3: a simple reduction eigenvalue method

Given a number k and a tolerance ε for the converged eigenvectors,

Step 1: use Algorithm 2 to execute k steps of the Arnoldi process;

Step 2: formulate the projected upper Hessenberg matrix H_k and derive all its eigenpairs;

Step 3: use equation (22) to check the converged eigenvectors for the eigensolution of the original eigenvalue problem;

Step 4: store the converged eigenpairs and stop.

There is no iterative property in Algorithm 3. It can only tell which eigenpairs have converged to the given required precision in a single reduction. The key question is how to make the simple reduction a reliable method for finding eigenvalues without missing any required ones, which will be discussed in section 3.3.

3.1. AN ARNOLDI ALGORITHM FOR SKEW-DEFINITE EIGENVALUE PROBLEMS

For undamped structural eigenvalue problems and undamped gyroscopic eigenvalue problems, matrix A is skew-symmetric. For these two cases, the corresponding eigenvalue problem (7) is in skew-definite form. By equations (17) and (18), the upper Hessenberg matrix H_k must be skew-symmetric and therefore has a very simple structure [23]:

$$H_k = \begin{bmatrix} 0 & -\beta_2 & & & \\ \beta_2 & 0 & -\beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{k-1} & 0 & -\beta_k \\ & & & \beta_k & 0 \end{bmatrix}. \tag{23}$$

By using equation (23), the reduction process can be simplified as follows.

Algorithm 4 [23]: an Arnoldi reduction process for undamped gyroscopic problems

Step 1: $u_1 \in R^{2n \times 1}$ be an arbitrary vector, satisfying $\| u_1 \| = (u_1^{\mathrm{T}} B u_1)^{1/2} = 1$;

Step 2: new Arnoldi vectors are generated by

$$\beta_{k+1} u_{k+1} = r_k = B^{-1} A u_k - \beta_{k-1} u_{k-1}, \quad \beta_{k+1} = (r_k^{\mathrm{T}} B r_k)^{1/2}, \qquad k = 1, \ldots. \tag{24}$$

This algorithm is even simpler than the symmetric Lanczos method, and is similar to Bauchau's method [12] except that it is conducted entirely in the real domain. Although the gyroscopic effects are present in the undamped gyroscopic system, the corresponding eigenvalue problem behaves in the same way as that in the undamped structural system. This is due to both systems being energy conserving. Of course, the eigenvalue problems for undamped structural systems thus can be determined as a generalized symmetric eigenvalue problem in the configuration space instead of in the state space.

3.2. DETERMINATION OF MULTIPLE EIGENVALUES

Multiple eigenvalues may be present in the eigenvalue problems for engineering structures which possess some types of geometric symmetry. In theoretical computation, the multiple eigenvalues can be determined only one multiple by one multiple through the breakdown of the continuation of the Lanczos process or the Arnoldi process. In finite precision implementation of Lanczos' process and Arnoldi's process, breakdown very rarely happens even if the system does have multiple eigenvalues. If no breakdown takes place in the reduction process, the projected tri-diagonal matrix in the Lanczos process and the upper Hessenberg matrix in the Arnoldi process will not be reduced. An unreduced tri-diagonal or upper Hessenberg matrix cannot have multiple eigenvalues [24]. This indicates that multiple eigenvalues must be determined as approximately equal eigenvalues in practical computation by reduction methods. Attempts had been made to circumvent this difficulty by adopting the block Lanczos methods or block Arnoldi method [15]; see the summary of this problem in reference [24]. However, any eigenvalue problems, when expressed in the computer through floating point expressions, are perturbed within round-off error. In other words, the symmetry of the original problem cannot be precisely retained by a finite precision arithmetic. No matter which numerical method is used, the eigenvalue problems are subject to perturbation. The block Lanczos method or the block Arnoldi method has advantages in parallel implementations and in using out-of-core memory [25]. Generally, the reduction method can determine the multiple eigenvalues to a precision which is much higher than that required in engineering.

3.3. FEASIBILITY OF REDUCTION METHOD FOR EIGENVALUE PROBLEMS IN ENGINEERING

The eigensolution of a reduction matrix H_k can only indicate which eigenvalues have converged to the required precision, but it cannot give any information about whether or not all of the required eigenvalues have converged, especially when there are multiple eigenvalues. This is why this method has a reputation for missing eigenvalues. Without a reliable technique to check and determine missing eigenpairs, the method cannot be used with confidence.

Parlett and Scott [10] used a restart technique in the reduction process as a criterion to stop the computation in the Lanczos algorithm for symmetric eigenvalue problems. The main idea of the restart technique is as follows. After a simple reduction process has been conducted, check and store the converged eigenpairs, then select a start vector or construct a start vector from the unconverged Ritz vectors. The start vector and the new generated Lanczos vector are orthogonalized with respect to the converged eigenvectors. In other words, the Krylov subspaces in the restart process are generated in the complementary invariant subspace of the invariant subspace spanned by the converged eigenvectors in the previous reduction processes. The Krylov subspaces can also be regarded as generated by the normal reduction process, with a matrix which is equal to the original matrix except that the converged eigenvalues are set to zero in the spectrum of the original matrix. As pointed out in reference [10], the inner eigenvalues may converge first before and eigenvalues in very specially constructed cases in the reduction process. In the restart process, the converged eigenvalues are removed from the end of the spectrum, and the inner eigenvalues within the vicinity of the end eigenvalues become essential end eigenvalues. Those eigenvalues will be the eigenvalues to converge in the restart process according to the Lanczos phenomenon. When a number (N_f) of eigenvalues or all the eigenvalues under a given threshold are required, then, if the number of all converged eigenvalues in the previous reduction is larger than N_f, and all the converged eigenvalues in the current restart process are not in the convex of the first m converged eigenvalues, or if all the converged eigenvalues in the current restart are not below the threshold, the whole solution procedure can be stopped; otherwise, a new restart process is required. Through this analysis, one can conclude that the reliability of the restart technique being used as a technique to check and to determine the missed eigenvalues is of the same reliability as the Lanczos method being used to determine the end eigenvalues.

A similar restart technique can also be used with Arnoldi's method for unsymmetric eigenvalue problems. The difference is that the bi-orthogonality of the right and left eigenvectors requires that the new generated Arnoldi vectors need to be orthogonalized with respect to the left eigenvectors corresponding to the converged eigenvalues. However, an unsymmetric eigenvalue problem may present complex eigenpairs. If the new generated Arnoldi vectors are directly orthogonalized with respect to complex eigenvectors, complex computation is inevitable. Of course, the complex computation can be avoided by separating the new generated Arnoldi vectors into the real and imaginary part corresponding to the complex eigenpairs. However, the parallel determination of the left and right eigenvalue problem has to be carried out simultaneously. Furthermore, difficulty may also arise in determining the eigenpairs for unsymmetric eigenvalue problems which have defective eigenvalues, and usually the defects of an unsymmetric eigenvalue problem are not known in advance. Therefore, one needs a unified method which is applicable to both defective and non-defective eigenvalue problems in the same way, and at best no complex computation is needed. Instead of using the left and right eigenvectors, the dynamic orthogonal decoupling technique [26, 27] has been adopted, and a restart technique for the unsymmetric eigenvalue problems without generating the left Krylov

subspace and computing the left eigenvectors has been given in reference [28]. The method in reference [28] has the virtue of being suitable for both the defective and non-defective eigenvalue problems, and complex computation has been completely avoided. The method in reference [28] is similar to the deflated iterative Arnoldi method [21] and also implies a new development in the Arnoldi reduction. When a number (N_f) of the lower eigenvalues or all the eigenvalues below a threshold frequency are required, a restart Arnoldi reduction eigenvalue method generally takes the following shape.

Algorithm 5: a restart reduction eigenvalue method

Set $r = 0$.

Step 1: carry out a simple reduction process by Algorithm 3, and store the orthogonal bases of the invariant subspace spanned by converged eigenpairs in the current reduction;

Step 2: randomly generate a start vector or construct a start vector as the linear combination of the unconverged Ritz vectors in the last reduction, and the start vector is normalized and orthogonalized to the converged invariant subspace, then carry out the rth restart reduction process by Algorithm 3, except that all the new generated vectors which are orthogonalized to the converged invariant subspace;

Step 3: if the number of the converged eigenvalues is still less than N_f or one of the converged eigenvalues in current reductions is still below the threshold frequency, go to Step 4; otherwise, (1) in the case that a specified number of eigenvalues is required: if none of the converged eigenvalues in the current reduction has the frequency which is below the highest frequency of the first N_f converged eigenvalues, the N_f converged eigenvalues and the corresponding eigenvectors are accepted as the required eigenpairs, then, go to Step 6, or else go to Step 4; (2) in the case that all the eigenvalues below a threshold frequency are required; if none of the converged eigenvalues in the current reduction has a frequency below the threshold frequency, the converged eigenvalues and the corresponding eigenvectors are the required eigensolution of the original damped eigenvalue problem, then, go to Step 6, otherwise go to Step 4;

Step 4: update the orthogonal bases of the invariant subspace spanned by the converged eigenvectors with the converged eigenvectors in the rth restart process;

Step 5: set $r = r + 1$ and go to Step 2;

Step 6: store the converged eigenvalues and the corresponding orthogonal bases of the invariant spaces. Stop.

Algorithm 5 can also be made safer in practical implementation. For example, in the case that N_f lower eigenvalues are required, the stop criterion can be modified as follows: the restart process is stopped if all the converged eigenvalues in the current reduction have a good clearance to the converged N_f eigenvalues. This modification is favorable when the last of the required N_f eigenvalues are multiple eigenvalues.

4. NUMERICAL IMPLEMENTATION AND EXAMPLES

4.1. THE EXPLOITATION OF THE SPARSITY OF THE SYSTEM MATRIX

The system matrix enters the algorithm only in the form of the following matrix–vector product:

$$B^{-1}Au_k = \begin{bmatrix} K^{-1} & \\ & M^{-1} \end{bmatrix} \begin{bmatrix} -(G+C) & -M \\ M & 0 \end{bmatrix} \begin{Bmatrix} q_k \\ p_k \end{Bmatrix}, \tag{25}$$

or

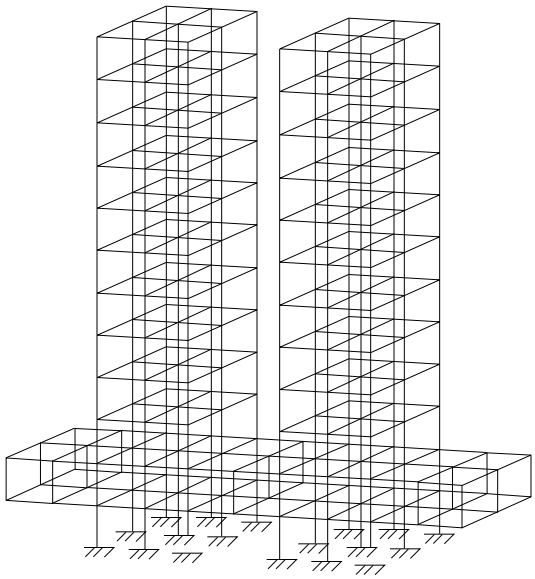
$$B^{-1}\mathrm{Au_k}=\begin{Bmatrix}-(L_kL_k^{\mathrm{T}})^{-1}[(Gq_k+Cq_k)+Mp_k]\\ q_k\end{Bmatrix}. \tag{26}$$

It can be determined with the sparsity of matrix M, G, C and K fully retained. The sparsity of K can also be fully retained in L_k by Cholesky decomposition with the one-dimensional variable bandwidth storing technique.

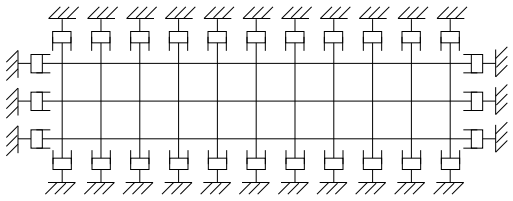
4.2. NUMERICAL EXAMPLES

4.2.1. *A double tower structure*

A double tower structure illustrated in Figure 1(a) was analyzed with each beam member of unit length. The material and cross-sectional properties were $E = 2{\cdot}1 \times 10^{11}$, $v = 0{\cdot}3$, $J_y = J_z = 8{\cdot}33 \times 10^{-6}$, $a = 1{\cdot}0 \times 10^{-2}$ and $\rho_l = 9{\cdot}8 \times 10^{1}$. Two layers of the foundation were damped as shown in Figure 1(b). The damping coefficient for each damper is $c = 1{\cdot}0 \times 10^{2}$. The computed results are shown in Tables 1–3. All the results were computed on an IBM PC 80486DX2-66 compatible. 45 Arnoldi vectors were generated for each reduction. The precision of the converged eigenpairs was taken to be $\varepsilon = 1{\cdot}0 \times 10^{-10}$, with reference to equation (22). The simultaneous iteration method [29] was also applied to this problem: the same results are obtained for all the three cases.



(a)



(b)

Figure 1. (a) A double tower structure. (b) Damping from a two-layer foundation.

TABLE 1

Undamped eigenvalues

$\pm$i23·1893	$\pm$i23·6953	$\pm$i23·7725 $\pm$ i26·6934	$\pm$i32·5076	$\pm$i33·7160

TABLE 2

Proportional damping: $C = 10 \times M$

$-5{\cdot}00000 \pm$ i22·6439	$-5{\cdot}00000 \pm$ i23·1618	$-5{\cdot}00000 \pm$ i23·2407
$-5{\cdot}00000 \pm$ i26·2209	$-5{\cdot}00000 \pm$ i32·1207	$-5{\cdot}00000 \pm$ i33·3432

TABLE 3

Concentratedly damped eigenvalues

$-0{\cdot}159980 \times 10^{-2} \pm$ i23·1893	$-0{\cdot}384710 \times 10^{-3} \pm$ i23·6953	$-0{\cdot}193554 \times 10^{-2} \pm$ i23·7725
$-0{\cdot}221652 \times 10^{-6} \pm$ i26·6934	$-0{\cdot}188746 \times 10^{-2} \pm$ i32·5076	$-0{\cdot}274793 \times 10^{-3} \pm$ i33·7160

Because there are no multiple eigenvalues in this example and the lower eigenvalues are also well separated, there are no missing eigenvalues among the converged ones; thus no restart was applied.

The computing times for the three cases were 6 minutes and 37 seconds, 6 minutes and 40 seconds, and 6 minutes and 35 seconds, respectively.

4.2.2. *A helicopter rotary wing system model*

An analysis was carried out for the helicopter rotary wing model illustrated in Figure 2. The material and cross-sectional parameters of blades were $E = 2{\cdot}1 \times 10^{11}$, $v = 0{\cdot}3$, $a = 2{\cdot}231 \times 10^{-5}$, $\rho_l = 1{\cdot}13 \times 10^{-1}$, $J_y = 1{\cdot}167 \times 10^{-11}$ and $J_z = 1{\cdot}167 \times 10^{-8}$. The blade length was $l_b = 0{\cdot}508$, $E = 2{\cdot}1 \times 10^{11}$, $v = 0{\cdot}3$, $a = 2{\cdot}231 \times 10^{-5}$, $\rho_l = 1{\cdot}13 \times 10^{-1}$ and $J_y = J_z = 1{\cdot}167 \times 10^{-8}$, the shaft length was $l_s = 0{\cdot}2$, and the distance from the damper to

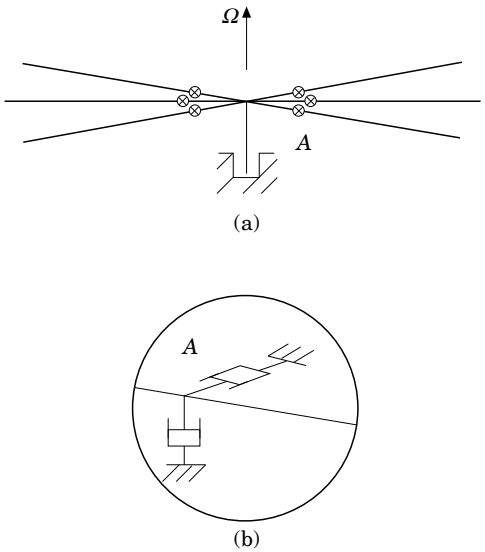


Figure 2. (a) A model of a helicopter rotary wing system. (b) Damper links.

TABLE 4

(a) *the undamped gyroscopic eigenvalues*

±i80·0583	±i80·0619	±i80·3409	±i80·3691	±i80·3691	±i80·3691
±i493·713	±i493·795	±i494·591	±i494·746	±i494·746	±i494·746
±i541·727	±i1217·57	±i1225·32	±i1229·27	±i1230·09	±i1230·09
±i1230·09	±i1320·32				

(b) *computing times for the undamped gyroscopic example*

Solution number	Size of reduced system	Number of restarts	Computer time
1	80	3	2 minutes 17 seconds
2	396 (full reduction)	0	8 minutes 9 seconds

the blade root was $l_c = 0{\cdot}101$. The damping coefficient for each damper was $c = 1{\cdot}0 \times 10^1$. The rotational speed was $\Omega = 126{\cdot}6$. For current computation, a 198-DOF finite element model was used. The first 20 pairs of eigenvalues were computed for the undamped and damped cases by the proposed restart reduction method in Algorithm 5.

For the undamped case, the simplification in section 3.1 was incorporated into the algorithm. The required precision for the eigenpairs was set to $\varepsilon = 1{\cdot}0 \times 10^{-15}$ and 80 Arnoldi vectors were generated in each reduction. The stop condition was satisfied after three restarts. In order to verify the results, a full reduction was also carried out in the undamped case. Both solutions give the same result for the required 20 eigenpairs; see Table 4(a). The comparison of the two solution processes is shown in Table 4(b).

To obtain the first 20 pairs of eigenvalues for the damped gyroscopic eigenvalue problem, 60, 80 and 396 vector reductions were used. The convergence precision for the eigenpairs was set to be $\varepsilon = 1{\cdot}0 \times 10^{-15}$. All three solutions gave the same results for the first 20 pairs of eigenvalues; see Table 5(a)—396 vectors were in a full reduction. For the full reduction, the projected upper Hessenberg H_{2n} is orthogonally similar to $B^{-1}A$; hence no restart is needed and the solution is equivalent to the application of the QR method directly to $B^{-1}A$. The computer times and other details for the three solution processes are shown in Table 5(b).

From Table 5(b) it can be seen that the reduction eigenvalue method has a great advantage in computing time compared with an eigenvalue method which finds the full solution. The computer times for the two full solutions for the undamped and damped case have a very large difference. This is due to the computation cost being very low with the simplified algorithm, both in the reduction process and in the eigensolution of the

TABLE 5

(a) *damped gyroscopic eigenvalues*

−2·53070 ± i80·1299	−2·53031 ± i80·1335	−2·49506 ± i80·4116	−2·48840 ± i80·4393
−2·48840 ± i80·4393	−2·48840 ± i80·4393	−32·8800 ± i497·398	−32·8455 ± i497·479
−32·5034 ± i498·235	−32·3951 ± i498·365	−32·3951 ± i498·365	−32·3951 ± i498·365
−9·48749 ± i541·657	−128·830 ± i1227·05	−133·192 ± i1231·50	−132·583 ± i1235·91
−132·067 ± i1236·76	−132·067 ± i1236·76	−132·067 ± i1236·76	−29·8358 ± i1317·02

(b) *computer times for the damped gyroscopic examples*

Solution number	Size of reduced system	Number of restarts	Computer times
1	60	3	4 minutes 2 seconds
2	80	3	4 minutes 40 seconds
3	396 (full reduction)	0	138 minutes

projected tri-diagonal matrix. From the theoretical results in reference [23], each root fixed eigenvalue of the single blade is a three-multiple eigenvalue of the rotary wing system with six blades. It can be seen from Tables 4(a) and 5(a) that the multiple eigenvalues were safely computed. There are new theoretical development and implementation details on the Arnoldi reduction and restart technique for the unsymmetric-definite problem in structural dynamics in reference [28].

5. CONCLUSIONS

Arnoldi's method has been generalized to solve large scale unsymmetric eigenvalue problems in structural dynamics. Advantages of Arnoldi's method which are retained in the proposed method are: (a) reduction process is performed entirely in real arithmetic; (b) no serious breakdown takes place in the reduction process irrespective of which system it is applied to; and (c) the left Krylov subspace or the left eigenvectors do not need to be computed, hence providing economy in computation. By enforcing the special orthogonality among the generalized Arnoldi vectors for the unsymmetric-definite problem, the symmetry of the system matrices can be retained in the projected upper Hessenberg matrix. By taking advantage of the retention of symmetry, a very simple reduction algorithm was obtained for undamped gyroscopic eigenvalue problems. This simple algorithm has a computation cost similar to that of Bauchau's gyroscopic Lanczos method, but has the advantage that the whole process is conducted entirely in the real domain. Since no guarantee is offered as to whether or not eigenpairs have been missed by a simple reduction method, the restart technique becomes an efficient way of checking and determining those possible missing eigenpairs. The reliability of the restart technique being used as a technique to check and determine the missed eigenvalues is the same as that of the Lanczos method being used to determine the end eigenvalues. Without such a missing eigenvalue check, a reduction method will not be safe in application to engineering problems. The numerical examples presented have demonstrated the potential of the proposed reduction method for large scale unsymmetric eigenvalue problems in structural dynamics.

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APPENDIX A

Lemma 1. For $P \in R^{n \times n}$, $\{q_1, \ldots, q_j\}$ are the Arnoldi vectors obtained by performing the following recurrence scheme on matrix P:

$$\beta_{i+1} q_{i+1} = r_i = P q_i - \sum_{k=1}^{i} h_{ki} q_k, \qquad i = 1, \ldots, j-1. \tag{A1}$$

Here $\| q_1 \| = 1$, $h_{ki} = q_k^{\mathrm{T}} P q_i$ and $\beta_{i+1} = \| r_i \|$. If there is no breakdown until step j, Arnoldi vectors $\{q_1, \ldots, q_j\}$ are a set of orthogonal bases for the Krylov subspace span $\{q_1, Pq_1, P^2 q_1, \ldots, P^{j-1} q_j\}$: i.e.,

$$\text{span}\,\{q_1, q_2, \ldots, q_j\} = \text{span}\,\{q_1, Pq_1, \ldots, P^{j-1} q_1\}. \tag{A2}$$

Proof. The conclusion is obviously correct for $j = 1$. Assume that the conclusion holds for $j = i$: i.e.,

$$\text{span}\,\{q_1, q_2, \ldots, q_i\} = \text{span}\,\{q_1, Pq_1, \ldots, P^{i-1} q_1\}. \tag{A3}$$

From equation (A3), q_i can be expressed as

$$q_i = \sum_{k=1}^{i} (P^{k-1} q_1)\alpha_k, \tag{A4}$$

and at least one of α_k $(k = 1, \ldots, i)$ is not equal to zero, so that

$$Pq_i = \sum_{k=1}^{i} (P^k q_1)\alpha_k \in \text{span}\,\{q_1, Pq_1, \ldots, P^i q_1\}. \tag{A5}$$

Also, from equation (A3),

$$\sum_{k=1}^{i} h_{ki} q_k \in \{q_1, \ldots, q_i\} \in \text{span}\,\{q_1, Pq_1, \ldots, P^{i-1} q_1\}. \tag{A6}$$

From the assumed condition in the lemma, $\beta_{i+1} \neq 0$, and combining equations (A1), (A3) and (A4) yields

$$q_{i+1} = \frac{1}{\beta_{i+1}} \left(Pq_i - \sum_{k=1}^{i} h_{ki} q_k \right) \in \text{span}\,\{q_1, Pq_1, \ldots, A^i q_1\}. \tag{A7}$$

From equation (A3), one has

$$P^{i-1} q_1 = \sum_{k=1}^{i} \gamma_k q_k. \tag{A8}$$

Pre-multiply equation (A8) with P to obtain

$$P^i q_1 = \sum_{k=1}^{i} \gamma_k P q_k, \tag{A9}$$

and recall the recurrence formulation (A1), to yield

$$P^i q_1 = \sum_{k=1}^{i} \gamma_k \left(\beta_{k+1} q_{k+1} + \sum_{s=1}^{k} h_{sk} q_s \right) \in \text{span}\,\{q_1, q_2, \ldots, q_{i+1}\}. \tag{A10}$$

Hence, by equations (A7) and (A10), the assumption is valid.

Proof of Theorem 1 [21]. From the assumption,

$$P^m q_1 = \sum_{i=1}^{k} s_i P^m y_i = \sum_{i=1}^{k} s_i \lambda_i^m y_i, \qquad m = 1, \ldots, j-1,$$

where $P = B^{-1}A$. Hence

$$\{q_1, Pq_1, \ldots, P^{j-1}q_1\} = [y_1 \quad \cdots \quad y_k]\begin{bmatrix} s_1 & \lambda_1 s_1 & \cdots & \lambda_1^{j-1}s_1 \\ \cdots & \cdots & \cdots & \cdots \\ s_k & \lambda_k s_k & \cdots & \lambda_k^{j-1}s_k \end{bmatrix} = [y_1 \quad \cdots \quad y_k]V, \tag{A11}$$

$$\text{rank}\,(q_1 \quad Pq_1 \quad \cdots \quad P^{j-1}q_1) = \text{rank}\,([y_1 \quad \cdots \quad y_k]V) = \begin{Bmatrix} j, & j \leqslant k \\ k, & j > k \end{Bmatrix}. \tag{A12}$$

From equation (A12), when $j < k$, $\{q_1, Pq_1, \ldots, P^{j-1}q_1\}$ is linearly independent; if $j > k$, then $\{q_1, Pq_1, \ldots, P^{j-1}q_1\}$ is linearly dependent. For $j = k + 1$,

$$P^k q_1 \in \text{span}\,\{q_1, Pq_1, \ldots, P^{k-1}q_1\} = \text{span}\,\{q_1, q_2, \ldots, q_k\}.$$

If $q_{k+1} \neq 0$ is the recurrence scheme obtained, from Lemma 1,

$$\text{span}\,\{q_1, q_2, \ldots, q_j\} = \text{span}\,\{q_1, Pq_1, \ldots, P^{j-1}q_1\}. \tag{A13}$$

From equation (A12), the dimensions of the two subspaces on the left and right sides of equation (A13) are k and $k + 1$, respectively. This is impossible. Hence the conclusion holds.

Theorem 2 is a direct application of Theorem 1.

APPENDIX B: NOMENCLATURE

A, B	matrices in state space	N_f	required number of eigenvalues
C	damping matrix, semi-positive definite	u_i	ith Arnoldi vector
D	circulatory matrix or constraint damping matrix, skew-symmetric	U_k	$=\{u_1, \ldots, u_k\}$, Arnoldi vector matrix
		a	cross-sectional area
E	Young's modulus of elasticity	n	order of matrices M, G, C, K and D
G	gyroscopic matrix, skew-symmetric	r	restart number
H_k	projected upper Hessenberg matrix in Arnoldi reduction process	s_i	ith eigenvector of matrix H_k
		s_{ki}	kth element of s_i
J_y, J_z	moments of beam cross-sectional area	x	displacement vector
K	stiffness matrix, symmetric	ε	tolerance given for determining convergence of eigenpairs
L_k	Cholesky triangle matrix in Cholesky factorization of matrix $K = L_k L_k^{\mathrm{T}}$	v	Poisson ratio
		ρ_l	mass density per unit length
M	mass matrix, positive definite		